Lecture 9 (1/24/22).

Recall. We showed that the collection of open sets (topology) of $\mathcal{C}(G,\Omega)$ does not depend on the exhaustion $\{K_n\}$ of G (like the metric), and neither do the convergent sequences: $f_k \to f$ in $\mathcal{C}(G,\Omega)$ iff $f_k \to f$ unif. on compacts $K \subset\subset G$.

The crucial point to establish these facts can be summarized in the following:

Lemma 1 • For any $\delta > 0$, $\exists\, K \subset\subset G$, $\varepsilon > 0$ s.t. $\sup_K d(f,g) < \varepsilon \Rightarrow \rho(f,g) < \delta$.

Conversely

• For any $K \subset\subset G$, $\varepsilon > 0$, $\exists\, \delta > 0$ s.t. $\rho(f,g) < \delta \Rightarrow \sup_K d(f,g) < \varepsilon$.

The proof of this is what gave the conclusions above about open sets and convergent seq.

Arzela-Ascoli Thm.

"set", "collection", ...

Def. ① A "family" $\mathcal{F}$ in $C(G,\Omega)$ is <u>normal</u> if every $\{f_k\}_{k=1}^{\infty} \subseteq \mathcal{F}$ has a convergent subsequence $\{f_{k_\ell}\}_{\ell=1}^{\infty}$ s.t. $f_{k_\ell} \to f \in C(G,\Omega)$.

Note: f need not belong to $\mathcal{F}$. Thus, normal $\Leftrightarrow \overline{\mathcal{F}}$ is compact.

Def. ② $\mathcal{F}$ is <u>equicontinuous</u> at $z_0 \in G$ if $\forall \varepsilon > 0\ \exists \delta > 0$ s.t. $\forall f \in \mathcal{F}$ and $|z - z_0| < \delta \Rightarrow d(f(z), f(z_0)) < \varepsilon$.

<u>Arzela-Ascoli Thm.</u> A family $\mathcal{F} \subseteq C(G,\Omega)$ is <u>normal $\Leftrightarrow$ TFH</u>:

(i) $\forall z \in G$, $\{f(z) : f \in \mathcal{F}\}$ compact.

(ii) $\mathcal{F}$ is equicont. at every $z \in G$.

For pf. of AA, we need a version of Tychonoff's Thm:

Let $\{(X_n, d_n)\}_{n=1}^{\infty}$ be collection of metric spaces. Consider product space (X, d), where $X = \prod_{n=1}^{\infty} X_n$ (i.e. an element $x \in X$ is $x = \{x_n\}_{n=1}^{\infty}$ w/ $x_n \in X$) and metric:

$$d(x,y) = \sum_{n=1}^{\infty} 2^{-n} \frac{d_n(x_n, y_n)}{1 + d_n(x_n, y_n)}.$$

Using similar arguments as in pf's above, one shows easily:

<u>Prop1.</u> A sequence $\{x^k\}_{k=1}^{\infty}$ in X converges to x $\Leftrightarrow$ each component seq. $\{x_n^k\}_{k=1}^{\infty}$ converges to x_n in X_n.

<u>Pf.</u> DIY.

__Tychonoff's Thm (special case).__ If each X_n is compact then $X = \prod_{1}^{\infty} X_n$ is compact.

Pf. One characterization of compactness: every seq. has a convergent subsequence (with limit in X). Let $\{x^k\}_{k=1}^{\infty}$ be seq. in X. Since X_1 is compact is a subseq. $k_j = k(1,j)$ s.t. $x_1^{k_j}$ converges to $x_1 \in X_1$. There is a subseq. of this $k(2,j)$ s.t. $x_2^{k(2,j)} \to x_2 \in X_2$. Continuing in this fashion we get nested subseq. $k(n,j)$ (subseq. of $k(n-1,j)$ etc.) s.t. $x_\ell^{k(n,j)} \xrightarrow[j\to\infty]{} x_\ell \in X_\ell$, $\ell \le n$.

Cantor's diagonal process: let $\{\tilde{x}^j\}_{j=1}^{\infty}$ be the subseq. s.t. the n:th component of $\tilde{x}_n^j = x_n^{k(j,j)}$. By constr. for $j \ge n$, $\tilde{x}_n^j$ is a subseq. of $x_n^{k(n,j)}$ and hence $\tilde{x}_n^j \to x_n \in X_n \Rightarrow \{\tilde{x}^j\}$ is conv. subseq. $\Rightarrow X$ is compact. ∎

For pf of Arzela-Ascoli Thm., we recall yet another characterization of compactness (Thm II. 4.9 in Conway):

- (X,d) is compact $\Leftrightarrow$ (X,d) is complete and totally bdd, i.e. $\forall \varepsilon>0$ $\exists x_1,\dots,x_N$ s.t. $X \subseteq \bigcup_{n=1}^{N} B(x_n,\varepsilon)$.

Pf of AA. Recall: $\mathcal{F}$ normal $\Leftrightarrow$ $\overline{\mathcal{F}}$ compact.

$\Rightarrow$. Show (i)+(ii) when $\overline{\mathcal{F}}$ compact:

(i). The evaluation map $E_z : \mathcal{C}(G,\Omega) \to \Omega$, $E_z(f) = f(z)$, is clearly continuous (by Lemma1) since $\{z\}$ is compact. $\Rightarrow$ $E_z(\overline{\mathcal{F}})$ is compact. Since $\overline{E_z(\mathcal{F})} = \overline{E_z(\mathcal{F})}$, (i) then follows.

(ii). Fix $z_0 \in G$ and pick $\varepsilon > 0$. Since $\overline{\mathcal{F}}$ is compact $\Rightarrow$ $\overline{\mathcal{F}}$ is totally bdd $\Rightarrow$ (HW) $\mathcal{F}$ is totally bdd. Let $\alpha_0 > 0$ s.t. $K = \overline{B(z_0;\alpha_0)} \subset\subset G$. By Lemma1,